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Limits of screened-Coulomb amplitudes

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Abstract. From two examples, one with an exponentially screened and the other with a smoothly cut-off Coulomb potential, arguments and partial proofs are given for the following conjecture. The absolute values of screened-Coulomb T matrices may converge in the limit of vanishing screening to the results of standard (i.e. short-range) theory. The condition is that the values of the momenta should be chosen before the limiting process.

1. Introduction

In recent years much work has been done to modify and extend scattering theory, in order to cover the Coulomb case (Dollard 1964, 1966, 1968, Prugovečki, 1971, 1973a, b, Zorbas 1974a, b, 1976, 1977). The main idea is to modify or 'renormalise' the wave operator by adding to the Hamiltonian certain time- and momentum-dependent terms which in a sense compensate the awkward long-range effects of the Coulomb potential.

When the standard short-range theory was used also for the long-range Coulomb scattering, expressions were obtained which however were defined only off the energy shells (Schwinger 1947, 1964, Okubo and Feldman 1960, Mapleton 1961, Hostler 1964). In order to remedy the situation, one introduced screenings of the Coulomb potential, and looked for possibly useful limits by vanishing screening (Dalitz 1951, Kacser 1959, Gorshkov 1961, Ford 1964, 1966, Rodberg and Thaler 1967). For a comprehensive survey, with numerous references, see Chen and Chen (1972).

It is the author's opinion, however, that these earlier investigations of Coulomb screening were not extended and completed towards the useful general conclusions which seem to be valid, and which do not seem to be sufficiently well known. Even if the aforementioned new renormalisation procedures exist, it would nevertheless be convenient to know exactly what information may be extracted from the standard theory for the limiting case of vanishing screening of the Coulomb potential.

In the present paper we make the following conjecture. The *absolute values* of screened-Coulomb T matrices may converge in the limit of vanishing screening to the results of standard short-range theory (see § 6). The condition is simply that the values of the momenta should be fixed *before* the limiting process. (This condition was in fact implicit in the treatment of Dalitz (1951), concerning the transition from the Yukawa to the Coulomb scattering amplitude. Later some limits of the cut-off Coulomb T matrix were evaluated by Ford (1964, 1966), who employed the abovementioned limiting procedure.) We shall treat in more detail these two examples, firstly an exponentially screened-Coulomb (i.e. Yukawa) potential, and secondly a

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smoothly cut-off Coulomb potential. The sharp cut-off, used for example by Ford, leads to certain extra problems (Semon and Taylor 1976), which disappear when a smooth 'tail' potential is added. Our method will be partly to give proofs for special cases of our assumption, and partly to check the assumption by employing old and new calculations to various orders.

2. Short-range potentials

For comparison and easy reference we present a short summary of the standard formulas, valid when $rV(r) \underset{r\to\infty}{\longrightarrow} 0$.

2.1. Total amplitudes

We shall use the amplitudes f, g and t, which are given respectively by ($\hbar = 1$):

$$f(\hat{\boldsymbol{r}}, \boldsymbol{k})\frac{1}{r}\,\mathrm{e}^{\mathrm{i}kr} + \mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{r}} \underset{\infty\leftarrow r}{\longleftarrow} \psi(\boldsymbol{r}, \boldsymbol{k}), \tag{2.1}$$

$$g(\boldsymbol{p}, \boldsymbol{k}) = -\frac{m}{2\pi}\int \mathrm{d}^3\boldsymbol{r}\;\mathrm{e}^{-\mathrm{i}\boldsymbol{p}.\boldsymbol{r}}V(r)\psi(\boldsymbol{r}, \boldsymbol{k}), \tag{2.2}$$

$$t(\boldsymbol{p}, \boldsymbol{p}'; k_\epsilon) = v(\boldsymbol{p}, \boldsymbol{p}') + \frac{1}{2\pi^2}\int \mathrm{d}^3\boldsymbol{q}\; v(\boldsymbol{p}, \boldsymbol{q})\frac{1}{q^2 - k_\epsilon^2}t(\boldsymbol{q}, \boldsymbol{p}'; k_\epsilon), \tag{2.3}$$

where $k_\epsilon = k + \mathrm{i}\epsilon \to k$, $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$, and

$$v(\boldsymbol{p}, \boldsymbol{p}') = -\frac{m}{2\pi}\int \mathrm{d}^3\boldsymbol{r}\;\mathrm{e}^{\mathrm{i}(\boldsymbol{p}'-\boldsymbol{p}).\boldsymbol{r}}V(r). \tag{2.4}$$

Equation (2.3) corresponds to the Lippmann–Schwinger equation $T = V + VG_0T$ for the transition operator T, with $t = -4\pi^2 mT$ and $v = -4\pi^2 mV$.

On the energy 'half'-shell and on the 'total' shell one has, respectively,

$$\boldsymbol{p}' = \boldsymbol{k}: \qquad t(\boldsymbol{p}, \boldsymbol{k}; k) = g(\boldsymbol{p}, \boldsymbol{k}), \tag{2.5a}$$

$$\boldsymbol{p}' = \boldsymbol{k}, p = k: \qquad t(\boldsymbol{p}, \boldsymbol{k}; k) = g(\boldsymbol{p}, \boldsymbol{k}) = f(\hat{\boldsymbol{p}}, \boldsymbol{k}). \tag{2.5b}$$

2.2. Partial amplitudes

The total amplitudes $a(=\psi, f, g, t, v)$ are expanded in partial 'waves' in the following way:

$$a(\boldsymbol{p}, \boldsymbol{q}) = \sum_{l=0}^{\infty} (2l+1)a_l(p, q)P_l(\hat{\boldsymbol{p}}.\hat{\boldsymbol{q}}). \tag{2.6}$$

For the partial amplitudes $a_l(=\psi_l, f_l, g_l, t_l, v_l)$ one then has respectively

$$\mathrm{i}^{-l}\psi_l(r, k) \equiv \phi_l(r, k) \underset{kr\gg 1}{\longrightarrow} \exp(\mathrm{i}\delta_l)\frac{1}{kr}\sin(kr - \tfrac{1}{2}l\pi + \delta_l), \tag{2.7}$$

$$f_l(k) = -2m \int_0^\infty \mathrm{d}r\, r^2 j_l(kr) V(r) \phi_l(r, k) \tag{2..8a}$$

$$= \frac{1}{2\mathrm{i}k}[\exp(2\mathrm{i}\delta_l) - 1], \tag{2.8b}$$

$$g_l(p, k) = -2m \int_0^\infty \mathrm{d}r\, r^2 j_l(pr) V(r) \phi_l(r, k), \tag{2.9}$$

$$v_l(p, p') = -2m \int_0^\infty \mathrm{d}r\, r^2 j_l(pr) V(r) j_l(p'r), \tag{2.10}$$

$$t_l(p, p'; k_\epsilon) = v_l(p, p') + \frac{2}{\pi} \int_0^\infty \mathrm{d}q\, q^2 v_l(p, q) \frac{1}{q^2 - k_\epsilon^2} t_l(q, p'; k_\epsilon), \tag{2.11}$$

$$p' = k: \qquad t_l(p, k; k) = g_l(p, k), \tag{2.12a}$$

$$p' = k = p: \qquad t_l(k, k; k) = g_l(k, k) = f_l(k). \tag{2.12b}$$

3. Coulomb potential

We write

$$V = V_c(r) = \frac{ZZ'e^2}{r} = -\frac{\nu k}{m}\frac{1}{r}, \qquad \nu = -\frac{ZZ'e^2 m}{k}. \tag{3.1}$$

3.1. Total amplitudes

The Coulomb scattering solution of the Schrödinger equation is

$$\psi_c(\boldsymbol{r}, \boldsymbol{k}) = \exp(\tfrac{1}{2}\pi\nu)\Gamma(1 - \mathrm{i}\nu) \exp(\mathrm{i}\boldsymbol{k}.\boldsymbol{r}) F(\mathrm{i}\nu, 1, \mathrm{i}(kr - \boldsymbol{k}.\boldsymbol{r})) \tag{3.2a}$$

$$\xrightarrow[kr - \boldsymbol{k}.\boldsymbol{r} \gg 1]{} \exp[-\mathrm{i}\nu \ln(2kr)] \exp(\mathrm{i}\boldsymbol{k}.\boldsymbol{r}) + \exp[+\mathrm{i}\nu \ln(2kr)] \frac{1}{r} \exp(\mathrm{i}kr) f_c(\hat{\boldsymbol{r}}, \boldsymbol{k}), \tag{3.2b}$$

where

$$f_c(\hat{\boldsymbol{r}}, \boldsymbol{k}) = \frac{2\nu k}{(\boldsymbol{p} - \boldsymbol{k})^2} \exp\left(\mathrm{i}\nu \ln \frac{(\boldsymbol{p} - \boldsymbol{k})^2}{4k^2} + 2\mathrm{i}\sigma_0\right), \tag{3.3a}$$

$$\text{with } \boldsymbol{p} = k\hat{\boldsymbol{r}} \text{ and } \sigma_0 = \arg \Gamma(1 - \mathrm{i}\nu). \tag{3.3b}$$

The modified form of the first factor in the first term of (3.2b) is obtained from the usual one because of the asymptotic form of the second factor, namely

$$\exp(\mathrm{i}kr\xi) \xrightarrow[kr \gg 1]{} \frac{\exp(\mathrm{i}kr)}{\mathrm{i}kr} \delta(1 - \xi) - \frac{\exp(-\mathrm{i}kr)}{\mathrm{i}kr} \delta(1 + \xi). \tag{3.4}$$

Hence only $\xi \equiv \hat{\boldsymbol{r}}.\hat{\boldsymbol{k}} = -1$ contributes, because $kr - \boldsymbol{k}.\boldsymbol{r} \gg 1$ requires that $\xi \neq +1$.

We emphasize that the Coulomb scattering amplitude f_c in (3.3a) cannot be obtained from an equation such as (2.2) with $p = k$, partly because ψ_c does not satisfy the usual integral equation.†

† A modified integral equation which is satisfied by ψ_c is given by the author (Kolsrud 1977). (See appendix (A.25)).

The amplitude g_c for $p \neq k$ exists, however, and is defined by

$$g_c(\boldsymbol{p}, \boldsymbol{k}) = \lim_{\epsilon \to 0} g_{c\epsilon}(\boldsymbol{p}, \boldsymbol{k}), \qquad p \neq k, \tag{3.5a}$$

where (Guth and Mullin 1951)

$$g_{c\epsilon}(\boldsymbol{p}, \boldsymbol{k}) = -\frac{m}{2\pi} \int \mathrm{d}^3\boldsymbol{r}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{p}.\boldsymbol{r}}\, \mathrm{e}^{-\epsilon r} V_c(r) \psi_c(\boldsymbol{r}, \boldsymbol{k}) \tag{3.5b}$$

$$= e^{\nu\pi/2} \Gamma(1 - \mathrm{i}\nu) \frac{2\nu k}{(\boldsymbol{p} - \boldsymbol{k})^2 + \epsilon^2} \left[\frac{(\boldsymbol{p} - \boldsymbol{k})^2 + \epsilon^2}{p^2 - (k + \mathrm{i}\epsilon)^2} \right]^{\mathrm{i}\nu}. \tag{3.5c}$$

Putting $p = k$ and comparing with (3.3), we get the 'anomalous' relation

$$g_{c\epsilon}(\boldsymbol{p}, \boldsymbol{k})_{p=k} \underset{\epsilon \approx 0}{\longrightarrow} \exp\left(\mathrm{i}\nu \ln \frac{2k}{\epsilon} \right) \Gamma(1 + \mathrm{i}\nu) f_c(\hat{\boldsymbol{p}}, \boldsymbol{k}). \tag{3.5d}$$

For later use, g_c is expanded in powers of ν. The lowest-order terms are

$$g_c^{(1)}(\boldsymbol{p}, \boldsymbol{k}) = \frac{2\nu k}{(\boldsymbol{p} - \boldsymbol{k})^2} = v_c(\boldsymbol{p}, \boldsymbol{k}),^{\dagger} \tag{3.6a}$$

$$g_c^{(2)}(\boldsymbol{p}, \boldsymbol{k}) = \frac{2\nu^2 k}{(\boldsymbol{p} - \boldsymbol{k})^2} (\mathrm{i}L + \mathrm{i}\gamma + \tfrac{1}{2}\pi), \tag{3.6b}$$

$$g_c^{(3)}(\boldsymbol{p}, \boldsymbol{k}) = \frac{2\nu^3 k}{(\boldsymbol{p} - \boldsymbol{k})^2} [-\tfrac{1}{2}L^2 + (\mathrm{i}\tfrac{1}{2}\pi - \gamma)L + \mathrm{i}\tfrac{1}{2}\pi\gamma - \tfrac{1}{2}\gamma^2 + \tfrac{1}{24}\pi^2], \tag{3.6c}$$

where

$$L = \lim_{\epsilon \to 0} \ln \frac{(\boldsymbol{p} - \boldsymbol{k})^2}{p^2 - k_\epsilon^2}, \tag{3.6d}$$

and γ is Euler's constant.

Even if the Coulomb case is not covered by the standard theory in § 2, there exists nevertheless a solution of the LS equation (2.3) with $v = v_c$, but only for $p \neq k \neq p'$, i.e. off the total energy shell. One of the many representations of this Coulomb t matrix is (cf Chen and Chen 1972)

$$t_c(\boldsymbol{p}, \boldsymbol{p}'; k_\epsilon) = \frac{2\nu k}{(\boldsymbol{p} - \boldsymbol{p}')^2} \left(1 - 4\mathrm{i}\nu \int_0^1 \frac{t^{-\mathrm{i}\nu}\, \mathrm{d}t}{\kappa(1 - t)^2 - 4t} \right), \tag{3.7a}$$

where

$$\kappa = \frac{(p^2 - k_\epsilon^2)(p'^2 - k_\epsilon^2)}{k^2(\boldsymbol{p} - \boldsymbol{p}')^2}. \tag{3.7b}$$

Expanding the transition matrix in (3.7a), we get

$$t_c^{(1)}(\boldsymbol{p}, \boldsymbol{p}'; \boldsymbol{k}) = \frac{2\nu k}{(\boldsymbol{p} - \boldsymbol{p}')^2} = v_c(\boldsymbol{p}, \boldsymbol{p}'), \tag{3.8a}$$

$$t_c^{(2)}(\boldsymbol{p}, \boldsymbol{p}'; k_\epsilon) = \frac{2\nu^2 k}{(\boldsymbol{p} - \boldsymbol{p}')^2} \mathrm{i}(1 + \kappa)^{-1/2} \ln \frac{1 + (1 + \kappa)^{1/2}}{1 - (1 + \kappa)^{1/2}} \tag{3.8b}$$

† Hence the Fourier transform (2.4) of $V_c(r)$ is defined by means of the convergence factor $\exp(-\epsilon r)$, as usual.

$$t_c^{(3)}(\boldsymbol{p}, \boldsymbol{p}'; k_\epsilon) = \frac{2\nu^3 k}{(\boldsymbol{p}-\boldsymbol{p}')^2}(1+\kappa)^{-1/2}(Li_2(t_+) - Li_2(t_-)), \tag{3.8c}$$

where

$$t_\pm = \frac{1}{\kappa}[2+\kappa \pm 2(1+\kappa)^{1/2}], \tag{3.8d}$$

and Euler's dilogarithm (Gröbner and Hofreiter 1950)

$$Li_2(z) = \int_0^1 \mathrm{d}t \frac{\ln t}{t - 1/z}. \tag{3.8e}$$

We observe that the Coulomb amplitudes f_c, g_c and t_c are defined in different regions of the (double) momentum space. The absolute values of g_c and t_c, however, exist in common regions. From (3.5*c*) it follows that

$$\lim_{\epsilon \to 0} |g_{c\epsilon}(\boldsymbol{p}, \boldsymbol{k})| = \frac{2|\nu| k}{(\boldsymbol{p}-\boldsymbol{k})^2} |\Gamma(1-\mathrm{i}\nu)| \times \begin{cases} \mathrm{e}^{\nu\pi/2}, & p > k, \\ 1, & p = k, \\ \mathrm{e}^{-\nu\pi/2}, & p < k. \end{cases} \tag{3.9}$$

According to formulae by Chen and Chen (1972) one gets

$$\lim_{\epsilon \to 0} \lim_{p' \to k} |t_c(\boldsymbol{p}, \boldsymbol{p}'; k_\epsilon)| = \frac{2|\nu| k}{(\boldsymbol{p}-\boldsymbol{k})^2} |\Gamma(1-\mathrm{i}\nu)|^2 \times \begin{cases} \mathrm{e}^{\nu\pi/2}, & p > k, \\ 1, & p = k, \\ \mathrm{e}^{-\nu\pi/2}, & p < k. \end{cases} \tag{3.10}$$

On the half-shell $p' = k$ we thus have

$$|t_c| : |g_c| = |\Gamma|^2 : |\Gamma|, \tag{3.11}$$

and on the total shell $p' = k = p$

$$|t_c| : |g_c| : |f_c| = |\Gamma|^2 : |\Gamma| : 1, \tag{3.12}$$

unlike the standard cases (2.5). The factors $|\Gamma|$ in (3.9) and $|\Gamma|^2$ in (3.10) constitute the Coulomb anomalies, in addition to the infinite phases (see e.g. (3.5*d*)).

3.2. *Partial amplitudes*

Expanding $\psi_c(\boldsymbol{r}, \boldsymbol{k})$ in (3.2*a*) in partial waves, like (2.6, 2.7), one has

$$\phi_{c,l}(r, k) = \mathrm{e}^{\nu\pi/2} \frac{\Gamma(l+1-\mathrm{i}\nu)}{(2l+1)!} \mathrm{e}^{\mathrm{i}kr}(2kr)^l F(l+1-\mathrm{i}\nu, 2l+2, -2\mathrm{i}kr) \tag{3.13a}$$

$$\xrightarrow[kr \gg 1]{} \exp(\mathrm{i}\sigma_l) \frac{1}{kr} \sin[kr - l\tfrac{1}{2}\pi + \nu \ln(2kr) + \sigma_l], \tag{3.13b}$$

where

$$\sigma_l = \arg \Gamma(l+1-\mathrm{i}\nu). \tag{3.13c}$$

Concerning an expansion like (2.6) of $f_c(\hat{\boldsymbol{p}}, \boldsymbol{k})$ in (3.3*a*), one sees that it cannot be inverted to give $f_{c,l}$ (leading to a divergent integral). We would rather choose the following procedure:

As shown in appendix equation (A.5) we obtain in the first place the series

$$f_c(\hat{\boldsymbol{p}}, \boldsymbol{k}) = \frac{1}{2ik} \sum_{l=1}^{\infty} [\exp(2i\sigma_{l-1}) - \exp(2i\sigma_{l+1})] P_l'(\xi), \tag{3.14}$$

where $\xi \equiv \hat{\boldsymbol{p}}.\hat{\boldsymbol{k}} \neq \pm 1$. (The lth term $\to O(l^{-1/2})$.) From the simple identity†

$$\sum_{l=1}^{\infty} (C_{l-1} - C_{l+1}) P_l'(\xi) = \sum_{l=0}^{\infty} C_l (2l+1) P_l(\xi) \tag{3.15}$$

we may *formally* write (3.14) as

$$f_c(\hat{\boldsymbol{p}}, \boldsymbol{k}) = \frac{1}{2ik} \sum_{l=0}^{\infty} [\exp(2i\sigma_l) - c](2l+1) P_l(\xi), \tag{3.16}$$

where c is arbitrary. With $c = 1$ we get the usual form

$$f_{c,l}(k) = \frac{1}{2ik} [\exp(2i\sigma_l) - 1]. \tag{3.17}$$

(The constant c does not contribute, as $\Sigma(2l+1)P_l(\xi) = 2\delta(\xi - 1) = 0$, because $\xi \neq 1$ in (3.2b).)

We note that the series (3.16) is divergent, (the lth term $\to O(l^{1/2})$) and should rather be considered as a distribution, according to Taylor 1974.

We should also remember that $f_{c,l}(k)$ cannot be obtained from $\phi_{c,l}(r, k)$ by an equation like (2.8a), by reason of the logarithmic phase in (3.13b).

However, the partial amplitude $g_{c,l}(p, k)$ exists for $p \neq k$, and is given by (cf (3.5))

$$g_{c,l}(p, k) = \lim_{\epsilon \to 0} g_{c\epsilon,l}(p, k), \qquad p \neq k, \tag{3.18a}$$

where—like (2.9)—

$$g_{c\epsilon,l}(p, k) = -2m \int_0^{\infty} dr\, r^2 j_l(pr)\, e^{-\epsilon r} V_c(r) \phi_{c,l}(r, k). \tag{3.18b}$$

As shown in appendix equations (A.12) and (A.13) we get when

$$p \neq k: \qquad g_{c,l}(p, k) = e^{\nu\pi/2} 2i\nu \Gamma(-l - i\nu) \left(-\frac{p}{k}\right)^l \mathrm{Im}\left(\frac{\partial^l}{\partial p^l} \frac{(p+k)^{l-i\nu}(p-k)^{l+i\nu}}{(p+q)^{l+1}}\right)_{q=p}, \tag{3.19a}$$

(also given by Ford 1964 for $l = 0$);

$$p = k: \qquad g_{c\epsilon,l}(k, k) \underset{\epsilon \approx 0}{\longrightarrow} |\Gamma(1 + i\nu)| \frac{1}{2ik} \left[\exp\left(2i\sigma_l - i\sigma_0 + i\nu \ln \frac{2k}{\epsilon}\right) - \exp\left(i\sigma_0 - i\nu \ln \frac{2k}{\epsilon}\right)\right]. \tag{3.19b}$$

To show the consistency with (3.5d), we rearrange the series (2.6) for $g_{c\epsilon}(\boldsymbol{p}, \boldsymbol{k})$ as in

† Use: $(2l+1)P_l(\xi) = P_{l+1}'(\xi) - P_{l-1}'(\xi)$.

(3.15), and get

$$g_{c\epsilon}(k\hat{\boldsymbol{p}}, \boldsymbol{k}) = \sum_{l=1} (g_{c\epsilon,l+1}(k,k) - g_{c\epsilon,l+1}(k,k))P_l'(\xi)$$

$$\underset{\epsilon\approx 0}{\longrightarrow} |\Gamma(1+i\nu)| \exp\left(-i\sigma_0 + i\nu \ln\frac{2k}{\epsilon}\right)$$

$$\times \frac{1}{2ik}\sum_{l=1} [\exp(2i\sigma_{l-1}) - \exp(2i\sigma_{l+1})]P_l'(\xi)$$

$$= \Gamma(1+i\nu)\exp\left(i\nu \ln\frac{2k}{\epsilon}\right) f_c(\hat{\boldsymbol{p}}, \boldsymbol{k}). \tag{3.20}$$

Concerning the partial-wave expansion of $t_c(\boldsymbol{p}, \boldsymbol{p}'; k_\epsilon)$, we show only how the first-order term $t_{c,l}^{(1)}$ behaves near the energy shell.† (Incidentally, from (3.10) it follows that the anomalous factor $|\Gamma|^2$ will not appear in $t_c^{(1)}$.) From (3.8*a*) we have

$$t_c^{(1)}(\boldsymbol{p}, \boldsymbol{p}'; k) = v_c(\boldsymbol{p}, \boldsymbol{p}') = \frac{2\nu k}{p^2 + p'^2 - 2pp'\xi}, \qquad \xi = \hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}', \tag{3.21}$$

giving

$$t_{c,l}^{(1)}(p, p'; k) = v_{c,l}(p, p') = \frac{\nu k}{pp'} Q_l\left(\frac{p^2+p'^2}{2pp'}\right), \qquad p \neq p', \tag{3.22a}$$

where

$$Q_l(z) = \frac{1}{2}\int_{-1}^{+1} d\xi \frac{P_l(\xi)}{z-\xi}, \qquad z > 1. \tag{3.22b}$$

As $Q_l(z)$ is divergent when $z \to 1$, we proceed in a different way when $p' \to p$. From appendix equation (A.7) we get the following series (when $|\xi| \neq 1$):

$$v_c(\boldsymbol{p}, \boldsymbol{p}')_{p'=p} = \frac{\nu k}{p^2}\frac{1}{1-\xi} = \frac{\nu k}{p^2}\sum_{l=1} (\psi(l+2) - \psi(l))P_l'(\xi), \tag{3.23a}$$

where

$$\psi(l) = \sum_{n=1}^{l-1}\frac{1}{n} - \gamma, \qquad \psi(1) = -\gamma. \tag{3.23b}$$

Using (3.15), we write formally

$$v_c(\boldsymbol{p}, \boldsymbol{p}')_{p'=p} = -\frac{\nu k}{p^2}\sum_{l=0} (2l+1)\psi(l+1)P_l(\xi), \tag{3.24a}$$

(which is divergent.) Hence

$$v_{c,l}(p,p) = -\frac{\nu k}{p^2}\psi(l+1), \tag{3.24b}$$

and

$$t_{c,l}^{(1)}(k,k;k) = v_{c,l}(k,k) = -\frac{\nu}{k}\psi(l+1). \tag{3.25}$$

† The complete expressions $t_{c,l}(p, p'; k_\epsilon)$ for $l = 0$ and $l = 1$ have been given by van Haeringen (1977).

This result agrees with (3.17), which has the first-order term

$$f^{(1)}_{c,l}(k)=\frac{1}{k}\sigma^{(1)}_l=-\frac{\nu}{k}\psi(l+1), \tag{3.26a}$$

because of (3.13c), and the definition (Magnus *et al* 1966)

$$\psi(z)=\Gamma'(z)/\Gamma(z). \tag{3.26b}$$

4. Exponentially screened-Coulomb potential

We consider the (Yukawa) potential

$$V=V_\mu(r)=e^{-\mu r}V_c(r)=-\frac{k\nu}{m}\frac{e^{-\mu r}}{r}\xrightarrow[\mu\to 0]{} V_c(r). \tag{4.1}$$

4.1. *Total amplitudes*

Gorshkov (1961) has shown that

$$\psi_\mu(\boldsymbol{r},\boldsymbol{k})\xrightarrow[\mu\approx 0]{}\exp(i\Delta_\mu)\psi_c(\boldsymbol{r},\boldsymbol{k}), \tag{4.2a}$$

where the diverging phase

$$\Delta_\mu=\nu\left(\ln\frac{2k}{\mu}-\gamma\right). \tag{4.2b}$$

For $\mu\neq 0$ the standard theory in § 2 is valid, so that f_μ and g_μ can be obtained from t_μ (cf (2.12)). We get

$$t^{(1)}_\mu(\boldsymbol{p},\boldsymbol{p}';k)=v_\mu(\boldsymbol{p},\boldsymbol{p}')=\frac{2k\nu}{(\boldsymbol{p}-\boldsymbol{p}')^2+\mu^2}. \tag{4.3}$$

The second- and third-order terms have been calculated by Gyland (1976), who obtained:

$$t^{(2)}_\mu(\boldsymbol{p},\boldsymbol{p}';k_\epsilon)=\frac{2k\nu^2}{R}i\ln\frac{A+R}{A-R}, \tag{4.4a}$$

where

$$A=(\boldsymbol{p}-\boldsymbol{p}')^2+i\frac{\mu}{k}(p^2+p'^2-2k^2_\epsilon)+4\mu^2+2i\frac{\mu^3}{k}, \tag{4.4b}$$

$$R^2=(\boldsymbol{p}-\boldsymbol{p}')^2\left((\boldsymbol{p}-\boldsymbol{p}')^2+\frac{1}{k^2}(p^2-k^2_\epsilon)(p'^2-k^2_\epsilon)\right)$$
$$+\frac{\mu^2}{k^2}[(\boldsymbol{p}-\boldsymbol{p}')^2(p^2+p'^2+2k^2)-(p^2-p'^2)^2]+\frac{\mu^4}{k^2}(\boldsymbol{p}-\boldsymbol{p}')^2. \tag{4.4c}$$

(These expressions with $\boldsymbol{p}'=\boldsymbol{k}$ and $p=k$ will—after some reductions—lead to $f^{(2)}_\mu$ for arbitrary μ, in the form given by, e.g., Dalitz (1951).)

The third-order term is given only for $p=0$:

$$t_\mu^{(3)}(0,\boldsymbol{p}';k_\epsilon)=2\nu^3\frac{k}{k^2+\mu^2}\cdot\frac{k}{p'}$$
$$\times\left[Li_2\left(\frac{(k+p'-\mathrm{i}\mu)(k-2\mathrm{i}\mu)}{(k_\epsilon-p'+\mathrm{i}\mu)(k+2\mathrm{i}\mu)}\right)-Li_2\left(\frac{(k_\epsilon-p'-\mathrm{i}\mu)(k-2\mathrm{i}\mu)}{(k+p'+\mathrm{i}\mu)(k+2\mathrm{i}\mu)}\right)\right.$$
$$\left.+Li_2\left(\frac{(k_\epsilon-p'-\mathrm{i}\mu)(-\mathrm{i}\mu)}{(k+p'+\mathrm{i}\mu)(2k+\mathrm{i}\mu)}\right)-Li_2\left(\frac{(k+p'-\mathrm{i}\mu)(-\mathrm{i}\mu)}{(k_\epsilon-p'+\mathrm{i}\mu)(2k+\mathrm{i}\mu)}\right)\right]. \tag{4.5}$$

(We have kept ϵ where zeros may occur.)

To obtain the screening limit it is important that the values of $\boldsymbol{p}$ and $\boldsymbol{p}'$ should be fixed (and $\epsilon\to 0$) *before* $\mu\to 0$. An exception is the first-order term (4.3), which for all $\boldsymbol{p}$ and $\boldsymbol{p}'$ becomes

$$t_\mu^{(1)}(\boldsymbol{p},\boldsymbol{p}';k)\underset{\mu\to 0}{\longrightarrow}v_c(\boldsymbol{p},\boldsymbol{p}')=t_c^{(1)}(\boldsymbol{p},\boldsymbol{p}';k). \tag{4.6}$$

Off both half-shells (4.4) and (4.5) yield when

$$p\neq k\neq p': \qquad t_\mu^{(2)}(\boldsymbol{p},\boldsymbol{p}';k)\underset{\mu\to 0}{\longrightarrow}\frac{2k\nu^2}{(\boldsymbol{p}-\boldsymbol{p}')^2}\mathrm{i}(1+\kappa)^{-1/2}\ln\frac{1+(1+\kappa)^{1/2}}{1-(1+\kappa)^{1/2}}$$
$$=t_c^{(2)}(\boldsymbol{p},\boldsymbol{p}';k)\qquad \text{in (3.8}b\text{)}, \tag{4.7}$$

$$p=0,\,p'\neq k: \qquad t_\mu^{(3)}(0,\boldsymbol{p}';k)\underset{\mu\to 0}{\longrightarrow}\frac{2\nu^3}{p'}\left[Li_2\left(\frac{k+p'}{k-p'}\right)-Li_2\left(\frac{k-p'}{k+p'}\right)\right]$$
$$=t_c^{(3)}(0,\boldsymbol{p}';k)\qquad \text{in (3.8}e\text{)}. \tag{4.8}$$

This agrees with the result of Gyland, who—by means of a variant of Gorshkov's technique—shows that off both half-shells the limit of t_μ for $\mu\to 0$ is given by a series with *finite* terms, which he sums to t_c.

For $\boldsymbol{p}'=\boldsymbol{k}$, $p\neq k$, i.e. on one half-shell, one may according to (2.2) and (2.5a) write

$$t_\mu(\boldsymbol{p},\boldsymbol{k};k)=-\frac{m}{2\pi}\int \mathrm{d}^3\boldsymbol{r}\,\exp(-\mathrm{i}\boldsymbol{p}.\boldsymbol{r})V_\mu(r)\psi_\mu(\boldsymbol{r},\boldsymbol{k}). \tag{4.9}$$

If we introduced the limiting form (4.2a) for ψ_μ in (4.9), it appears from (3.5) (with $\epsilon=\mu$) that we would get for

$$p\neq k: \qquad t_\mu(\boldsymbol{p},\boldsymbol{k};k)\underset{\mu\approx 0}{\longrightarrow}\exp(\mathrm{i}\Delta_\mu)g_c(\boldsymbol{p},\boldsymbol{k}). \tag{4.10}$$

This assumption can be checked in second and in third order, by choosing $\boldsymbol{p}'=\boldsymbol{k}$ in (4.4) and (4.5). We then get (for $p\neq k$):

$$\lim_{\substack{\boldsymbol{p}'\to\boldsymbol{k}\\ \epsilon\to 0}} t_\mu^{(2)}(0,\boldsymbol{p}';k_\epsilon)\underset{\mu\approx 0}{\longrightarrow}\frac{2k\nu^2}{(\boldsymbol{p}-\boldsymbol{k})^2}\left(\mathrm{i}\ln\frac{(\boldsymbol{p}-\boldsymbol{k})^2}{p^2-k^2}+\frac{\pi}{2}+\mathrm{i}\ln\frac{2k}{\mu}\right), \tag{4.11}$$

$$\lim_{\substack{\boldsymbol{p}'\to\boldsymbol{k}\\ \epsilon\to 0}} t_\mu^{(3)}(0,\boldsymbol{p}';k_\epsilon)\underset{\mu\approx 0}{\longrightarrow}\frac{2\nu^3}{k}\left[Li_2\left(\frac{2k}{\mathrm{i}\mu}\right)-Li_2(-1)\right]\to\frac{2\nu^3}{k}\left(\frac{\pi^2}{24}-\frac{1}{2}\left(\ln\frac{2k}{\mu}\right)^2-\mathrm{i}\frac{\pi}{2}\ln\frac{2k}{\mu}\right), \tag{4.12}$$

(cf Gröbner and Hofreiter 1950). Comparison with (3.6) reveals that in fact, for $p \neq k$,

$$\lim_{\substack{p'\to k\\ \epsilon\to 0}} t_\mu^{(2)}(\boldsymbol{p}, \boldsymbol{p}'; k_\epsilon) \underset{\mu\approx 0}{\longrightarrow} g_c^{(2)}(\boldsymbol{p}, \boldsymbol{k}) + \mathrm{i}\nu\left(\ln\frac{2k}{\mu} - \gamma\right) g_c^{(1)}(\boldsymbol{p}, \boldsymbol{k}), \tag{4.13}$$

$$\lim_{\substack{p'\to k\\ \epsilon\to 0}} t_\mu^{(3)}(0, \boldsymbol{p}'; k_\epsilon) \underset{\mu\approx 0}{\longrightarrow} g_c^{(3)}(0, \boldsymbol{k}) + \mathrm{i}\nu\left(\ln\frac{2k}{\mu} - \gamma\right) g_c^{(2)}(0, \boldsymbol{k}) + \frac{1}{2}\left[i\nu\left(\ln\frac{2k}{\mu} - \gamma\right)\right]^2 g_c^{(1)}(0, \boldsymbol{k}). \tag{4.14}$$

Hence (4.10) is confirmed to $O(\nu^3)$.

Notice the importance of choosing $\boldsymbol{p}'$ and $\boldsymbol{p}$ and making $\epsilon \to 0$ *before* we let $\mu \approx 0$. For example, (4.7) and (4.8) would of course not lead to (4.11) and (4.12).

For $\boldsymbol{p}' = \boldsymbol{k}$ and $p = k$, i.e. on both half-shells, one might further expect from (4.10) and (3.5d) (with $\epsilon = \mu$) that the value of

$$\lim_{\mu\approx 0} \lim_{\substack{p\to k\\ \epsilon\to 0}} t_\mu(\boldsymbol{p}, \boldsymbol{k}; k_\epsilon) \qquad [= \lim_{\mu\approx 0} f_\mu(\hat{\boldsymbol{p}}, \boldsymbol{k})] \tag{4.15a}$$

would be equal to

$$\exp(\mathrm{i}\Delta_\mu) \lim_{\mu\approx 0} \lim_{p\to k} g_{c\mu}(\boldsymbol{p}, \boldsymbol{k}) = \exp(\mathrm{i}\Delta_\mu)\Gamma(1+\mathrm{i}\nu)\exp\left(\mathrm{i}\nu \ln\frac{2k}{\mu}\right) f_c(\hat{\boldsymbol{p}}, \boldsymbol{k}). \tag{4.15b}$$

This, however, is not correct. As

$$|\Gamma(1+\mathrm{i}\nu)| = 1 - \frac{\nu^2\pi^2}{12} + O(\nu^3), \tag{4.16}$$

the tentative relation (4.15) can be disproved, and replaced, by calculating (4.15a) to third order. The first- and second-order terms are obtained from (4.3) and (4.4), while the third-order term is given by Kacser (1959). Collecting these results, we get on the total energy shell

$$\begin{aligned} t_\mu^{(1)} + t_\mu^{(2)} + t_\mu^{(3)} = f_\mu^{(1)} + f_\mu^{(2)} + f_\mu^{(3)} &\underset{\mu\approx 0}{\longrightarrow} \frac{2k\nu}{(\boldsymbol{p}-\boldsymbol{k})^2}\left[1 + \mathrm{i}\nu \ln\frac{(\boldsymbol{p}-\boldsymbol{k})^2}{\mu^2} + \frac{1}{2}\left(\mathrm{i}\nu \ln\frac{(\boldsymbol{p}-\boldsymbol{k})^2}{\mu^2}\right)^2\right] \\ &= \frac{2k\nu}{(\boldsymbol{p}-\boldsymbol{k})^2}\exp\left(\mathrm{i}\nu \ln\frac{(\boldsymbol{p}-\boldsymbol{k})^2}{\mu^2}\right) - O(\nu^4) \\ &= \frac{2k\nu}{(\boldsymbol{p}-\boldsymbol{k})^2}\exp\left(\mathrm{i}\nu \ln\frac{(\boldsymbol{p}-\boldsymbol{k})^2}{4k^2} + 2\mathrm{i}\sigma_0 - 2\mathrm{i}\nu\gamma + 2\mathrm{i}\nu \ln\frac{2k}{\mu}\right) + O(\nu^4), \end{aligned} \tag{4.17a}$$

because

$$\sigma_0 = \arg\Gamma(1-\mathrm{i}\nu) = \nu\gamma + O(\nu^3). \tag{4.17b}$$

Comparing with (3.3a) and (4.2b), we therefore assume that

$$\lim_{\substack{p\to k\\ \epsilon\to 0}} t_\mu(\boldsymbol{p}, \boldsymbol{k}; k_\epsilon) = f_\mu(\hat{\boldsymbol{p}}, \boldsymbol{k}) \underset{\mu\approx 0}{\longrightarrow} \exp(2\mathrm{i}\Delta_\mu) f_c(\hat{\boldsymbol{p}}, \boldsymbol{k}), \tag{4.18a}$$

rather than (4.15b). Hence

$$\lim_{\mu\to 0} |f_\mu(\hat{\boldsymbol{p}}, \boldsymbol{k})| = |f_c(\hat{\boldsymbol{p}}, \boldsymbol{k})|, \tag{4.18b}$$

which is Dalitz's (1951) conjecture. Note the doubling of the (diverging) phase in (4.18*a*), as compared with (4.2*a*). The necessity of letting $p \to p' \to k$ (and $\epsilon \to 0$) before $\mu \to 0$ has been pointed out by Gyland (1976).

In passing, it seems important to emphasise the different values of the two limits (3.9) (with $\epsilon = \mu$) and (4.18*b*). In spite of (4.2*a*) we get

$$\lim_{\mu\to 0}\lim_{p\to k}\left|\frac{\langle \boldsymbol{p}|V_\mu|\psi_c\rangle}{\langle \boldsymbol{p}|V_\mu|\psi_\mu\rangle}\right| = |\Gamma(1+\mathrm{i}\nu)|, \tag{4.19}$$

rather than 1. The reason must be (Gyland and Kolsrud 1976) that during the integration we use $\psi_\mu(\boldsymbol{r}, \boldsymbol{k})$ for $r \to \infty$ while still $\mu > 0$. Then ψ_μ behaves asymptotically in the standard way (2.1), in contrast to $\psi_c(\boldsymbol{r}, \boldsymbol{k})$, which behaves like (3.2*b*). As $p = k$, these asymptotic values contribute more to the integral than when $p \neq k$ as in (4.10).

4.2. *Partial amplitudes*

We consider only the first-order term $t^{(1)}_{\mu,l} = v_{\mu,l}$, and observe the behaviour near the energy shell. (Compare with (3.21–3.26).) From (4.3) we get

$$v_{\mu,l}(p, p') = \frac{\nu k}{pp'} Q_l\left(\frac{p^2 + p'^2 + \mu^2}{2pp'}\right) \underset{\mu\to 0}{\longrightarrow} v_c(p, p'). \tag{4.20}$$

Hence, when

$$p \neq p': \qquad t^{(1)}_{\mu,l}(p, p'; k) \underset{\mu\to 0}{\longrightarrow} t^{(1)}_{c,l}(p, p'; k). \tag{4.21}$$

When $p = p' = k$, we have

$$v_{\mu,l}(k, k) = \frac{\nu}{k} Q_l\left(1 + \frac{\mu^2}{2k^2}\right), \tag{4.22}$$

which can be expressed as (Magnus *et al* 1966)

$$t^{(1)}_{\mu,l}(k, k; k) = f^{(1)}_{\mu,l}(k) = v_{\mu,l}(k, k)$$
$$= \frac{\nu}{k}\left\{P_l\left(1 + \frac{\mu^2}{2k^2}\right)\left[\frac{1}{2}\ln\frac{2 + \mu^2/2k^2}{\mu^2/2k^2} - \sum_{n=1}^{l}\frac{1}{n}\right] + \text{polynomial in } \mu^2/2k^2\right\} \tag{4.23a}$$

$$\underset{\mu\approx 0}{\longrightarrow} \frac{\nu}{k}\left\{\ln\frac{2k}{\mu} - \gamma - \psi(l+1)\right\} = \frac{1}{k}\{\Delta_\mu + \sigma^{(1)}_l\}, \tag{4.23b}$$

according to (3.23*b*), (3.26) and (4.2*b*). This corresponds to (4.18*a*), which by means of (3.14)–(3.15) can be written

$$t_\mu(\boldsymbol{p}, \boldsymbol{k}; k) = f_\mu(\hat{\boldsymbol{p}}, \boldsymbol{k})$$
$$\underset{\mu\approx 0}{\longrightarrow} \frac{1}{2\mathrm{i}k}\sum_{l=1} \{\exp[2\mathrm{i}(\Delta_\mu + \sigma_{l-1})] - \exp[2\mathrm{i}(\Delta_\mu + \sigma_{l+1})]\}P'_l(\xi)$$
$$= \frac{1}{2\mathrm{i}k}\sum_{l=0} (2l+1)\{\exp[2\mathrm{i}(\Delta_\mu + \sigma_l)] - 1\}P_l(\xi). \tag{4.24a}$$

Hence

$$f_{\mu,l}(k) \underset{\mu\approx 0}{\longrightarrow} \frac{1}{2ik}\{\exp[2i(\Delta_\mu + \sigma_l)] - 1\}, \tag{4.24b}$$

the first-order term of which is (4.23*b*).

Note that in these first-order partial amplitudes it is important to specify p and p' before $\mu \to 0$. In the total amplitudes this was not necessary until the second order.

5. Cut-off Coulomb potential with tail

We shall use a 'smoothly' cut-off Coulomb potential, namely

$$V = V_R(r) = \begin{cases} V_c(r) \\ W_R(r) \end{cases} = -\frac{k\nu}{m} \times \begin{cases} 1/r, & r \leqslant R, \\ w_R(r), & r \geqslant R, \end{cases} \tag{5.1a}$$

where

$$W_R(R) = V_c(R), \qquad \text{i.e. } w_R(R) = 1/R. \tag{5.1b}$$

We also require that for $n = 1, 2, \ldots$

$$\frac{d^{n-1}}{dr^{n-1}}(rw_r(r)) \underset{r\to\infty}{\longrightarrow} 0, \qquad \frac{d^n}{dr^n}(rw_R(r)) \underset{R\to\infty}{\longrightarrow} 0. \tag{5.1c}$$

The reason for using the 'tail'-potential $W_R(r)$ is the following: with $\boldsymbol{q} = \boldsymbol{p}' - \boldsymbol{p}$ we get from (2.4) by partial integration

$$\begin{aligned} v_R(\boldsymbol{p}, \boldsymbol{p}') &= \frac{2k\nu}{q}\left(\int_0^R dr \sin qr + \int_R^\infty dr\, rw_R(r) \sin qr\right) \\ &= \frac{2k\nu}{q^2}\left(1 + \int_R^\infty dr (rw_R(r))' \cos qr\right) \underset{R\to\infty}{\longrightarrow} \frac{2k\nu}{q^2} = v_c(\boldsymbol{p}, \boldsymbol{p}'). \end{aligned} \tag{5.2}$$

With a 'sharp' cut-off, i.e. $W_R(r) = 0$, one would get

$$v_R^{\text{sharp}}(\boldsymbol{p}, \boldsymbol{p}') = \frac{2k\nu}{q^2}(1 - \cos qR), \tag{5.3}$$

which has no limit when $R \to \infty$ (cf Semon and Taylor 1976 for this problem).

5.1. *Wavefunctions and scattering amplitude*

The radial functions are (see (3.13)):

$$r \leq R: \qquad \phi_{R,l}^{<}(r, k) = a_l \phi_{c,l}(r, k) \tag{5.4a}$$

$$\underset{\substack{kr\gg 1 \\ (kR\gg 1)}}{\longrightarrow} a_l \exp(i\sigma_l) \frac{1}{kr} \sin\left(kr - l\frac{\pi}{2} + \nu \ln(2kr) + \sigma_l\right), \tag{5.4b}$$

$$r \geqslant R: \qquad \phi_{R,l}^{>}(r, k) \underset{kR\gg 1}{\longrightarrow}$$

$$a_l \exp(i\sigma_l) \frac{1}{kr} \sin\left(kr - l\frac{\pi}{2} + \nu \ln(2kR) + \sigma_l + \tau_R(r)\right), \tag{5.5a}$$

where the 'tail'-phase is given by

$$\tau_R(r)=\nu\int_R^r \mathrm{d}r\, w_R(r). \tag{5.5b}$$

The justification of (5.5a) is that the radial differential equations are satisfied when $R\to\infty$, because of (5.1b, c).

As $\phi^{>}_{R,l}(r,k)$ must have the standard form (2.7) when $r\to\infty$, (because $rV_R(r)\xrightarrow[r\to\infty]{}0$ for finite R), we see that

$$\delta_l=\sigma_l+\Delta_R, \tag{5.6a}$$

where

$$\Delta_R=\nu\ln(2kR)+\tau_R, \qquad \tau_R\equiv\tau_R(\infty)=\nu\int_R^\infty \mathrm{d}r\, w_R(r). \tag{5.6b}$$

It also follows from (2.7), (5.5) and (5.6) that

$$a_l=\exp(\mathrm{i}\Delta_R). \tag{5.7}$$

Because Δ_R is independent of l, we get for $r\leqslant R$, by summing the series (2.6) for ψ_c,

$$\psi^{<}_R(\boldsymbol{r},\boldsymbol{k})\xrightarrow[kR\gg1]{}\exp(\mathrm{i}\Delta_R)\psi_c(\boldsymbol{r},\boldsymbol{k}). \tag{5.8}$$

For $kr\geqslant kR\gg1$ the equations (5.5a), (5.6) and (5.7) show that the 'outer' function can be written

$$\psi^{>}_R(\boldsymbol{r},\boldsymbol{k})=\exp(\mathrm{i}\boldsymbol{k}.\boldsymbol{r}+\mathrm{i}\tau_R-\mathrm{i}\tau_R(r))+f_c(\hat{\boldsymbol{r}},\boldsymbol{k})\frac{1}{r}\exp[\mathrm{i}kr+2\mathrm{i}\nu\ln(2kR)+\mathrm{i}\tau_R+\mathrm{i}\tau_R(r)], \tag{5.9}$$

which is confirmed by using (3.4). Furthermore we get the $\boldsymbol{r}=\boldsymbol{R}$ (note that $\tau_R(R)=0$):

$$\psi^{>}_R(\boldsymbol{R},\boldsymbol{k})=\exp(\mathrm{i}\tau_R)\left(\exp(\mathrm{i}\boldsymbol{k}.\boldsymbol{R})+\exp[2\mathrm{i}\nu\ln(2kR)]f_c\frac{\exp(\mathrm{i}kR)}{R}\right), \tag{5.10}$$

which according to (5.8), (5.6b) and (3.2b) is equal to the inner function

$$\psi^{<}_R(\boldsymbol{R},\boldsymbol{k})=\exp[\mathrm{i}\nu\ln(2kR)+\mathrm{i}\tau_R]$$
$$\times\left(\exp[-\mathrm{i}\nu\ln(2kR)+\mathrm{i}\boldsymbol{k}.\boldsymbol{R}]+\exp[\mathrm{i}\nu\ln(2kR)]f_c\frac{\exp(\mathrm{i}kR)}{R}\right). \tag{5.11}$$

And lastly, (5.9) assumes the standard asymptotic form

$$\psi^{>}_R(\boldsymbol{r},\boldsymbol{k})\xrightarrow[r\to\infty]{}\exp(\mathrm{i}\boldsymbol{k}.\boldsymbol{r})+f_R(\hat{\boldsymbol{r}},\boldsymbol{k})\frac{\exp(\mathrm{i}kr)}{r}. \tag{5.12}$$

Recalling (5.6b), we see that

$$f_R(\hat{\boldsymbol{r}},\boldsymbol{k})=\exp(2\mathrm{i}\Delta_R)f_c(\hat{\boldsymbol{r}},\boldsymbol{k}). \tag{5.13}$$

Alternatively, we might use (5.6a) and rearrange the series for f_R, as in (3.15), to get

$$f_R(\hat{\boldsymbol{r}}, \boldsymbol{k}) = \frac{1}{2\mathrm{i}k} \sum_{l=1} \{\exp(2\mathrm{i}\delta_{l-1}) - \exp(2\mathrm{i}\delta_{l+1})\} P_l'(\xi)$$

$$= \exp(2\mathrm{i}\Delta_R) \frac{1}{2\mathrm{i}k} \sum_{l=1} \{\exp(2\mathrm{i}\sigma_{l-1}) - \exp(2\mathrm{i}\sigma_{l+1})\} P_l'(\xi)$$

$$= \exp(2\mathrm{i}\Delta_R) f_c(\hat{\boldsymbol{r}}, \boldsymbol{k}). \tag{5.14}$$

5.2. Total t-matrix

According to (2.3) t is given by v (e.g. by iteration). As $v_R \to v_c$ when $R \to \infty$, and $t_c(\boldsymbol{p}, \boldsymbol{p}'; k)$ exists for $p \neq k \neq p'$ we therefore expect that off both half-shells

$$\lim_{R\to\infty} t_R(\boldsymbol{p}, \boldsymbol{p}'; k) = t_c(\boldsymbol{p}, \boldsymbol{p}'; k), \tag{5.15}$$

like the earlier example (4.6)–(4.8).

For $\boldsymbol{p}' = \boldsymbol{k}$ and p arbitrary, § 2.1 tells that

$$t_R(\boldsymbol{p}, \boldsymbol{k}; k) = -\frac{m}{2\pi} \int \mathrm{d}^3\boldsymbol{r}\, \exp(-\mathrm{i}\boldsymbol{p}.\boldsymbol{r}) V_R(r) \psi_R(\boldsymbol{r}, \boldsymbol{k}), \tag{5.16}$$

$$t_R(\boldsymbol{p}, \boldsymbol{k}; k)_{p=k} = f_R(\hat{\boldsymbol{p}}, \boldsymbol{k}). \tag{5.17}$$

As $V_R(r) \to V_c(r)$ when $R \to \infty$, the equations (5.8), (3.5), (5.13) and (5.17) show that

$$t_R(\boldsymbol{p}, \boldsymbol{k}; k) \xrightarrow[R\to\infty]{} \begin{cases} \exp(\mathrm{i}\Delta_R) g_c(\boldsymbol{p}, \boldsymbol{k}), & p \neq k, \quad (5.18a) \\ \exp(2\mathrm{i}\Delta_R) f_c(\hat{\boldsymbol{p}}, \boldsymbol{k}), & p = k, \quad (5.18b) \end{cases}$$

like (4.10) and (4.18a), respectively. The reason why the integral (5.16) with a limiting expression for ψ_R now can be used also on the total energy shell ($p' = k = p$), in contrast to the Yukawa case (4.9), (4.15), etc, is believed to be the fact that we know the exact wavefunction $\psi_R(\boldsymbol{r}, \boldsymbol{k})$ for *finite* (but large) value of R, while $\psi_\mu(\boldsymbol{r}, \boldsymbol{k})$ was only known near the limit $\mu = 0$.

It seems therefore worthwhile also in this example to examine more closely the limiting processes in (5.18). As the integral (5.16) exists for finite R, it can be written as a limiting value, namely

$$t_R(\boldsymbol{p}, \boldsymbol{k}; k) = \lim_{\mu\to 0} I(\mu) \tag{5.19a}$$

where

$$I(\mu) = -\frac{m}{2\pi} \int_{(r<R)} \mathrm{d}^3\boldsymbol{r}\, \exp(-\mathrm{i}\boldsymbol{p}.\boldsymbol{r}) \exp(-\mu r) V_c(r) \psi_R^<(\boldsymbol{r}, \boldsymbol{k})$$

$$-\frac{m}{2\pi} \int_{(r>R)} \mathrm{d}^3\boldsymbol{r}\, \exp(-\mathrm{i}\boldsymbol{p}.\boldsymbol{r}) W_R(r) \psi_R^>(\boldsymbol{r}, \boldsymbol{k})$$

$$= -\frac{m}{2\pi} \exp(\mathrm{i}\Delta_R) \bigg[\int \mathrm{d}^3\boldsymbol{r}\, \exp(-\mathrm{i}\boldsymbol{p}.\boldsymbol{r} - \mu r) V_c(r) \psi_c(\boldsymbol{r}, \boldsymbol{k}) \qquad (= I_1)$$

$$- \int_{(r>R)} \mathrm{d}^3\boldsymbol{r}\, \exp(-\mathrm{i}\boldsymbol{p}.\boldsymbol{r} - \mu r) V_c(r) \psi_c(\boldsymbol{r}, \boldsymbol{k}) \bigg] \qquad (= I_2)$$

$$-\frac{m}{2\pi}\int_{(r>R)} d^3\boldsymbol{r}\, \exp(-i\boldsymbol{p}.\boldsymbol{r})W_R(r)\psi_R^{>}(\boldsymbol{r},\boldsymbol{k}). \qquad (=I_3) \tag{5.19b}$$

Recalling (3.5*a*)–(3.5*d*) (with $\epsilon=\mu$) we get

$$p\neq k: \qquad I_1 \xrightarrow[u\to 0]{} \exp(i\Delta_R)g_c(\boldsymbol{p},\boldsymbol{k}), \tag{5.20a}$$

$$p=k: \qquad I_1 \xrightarrow[u\approx 0]{} \exp(i\Delta_R)\left(\frac{2k}{\mu}\right)^{i\nu}\Gamma(1+i\nu)f_c(\hat{\boldsymbol{p}},\boldsymbol{k}). \tag{5.20b}$$

As shown in appendix 4 we get, with $\boldsymbol{q}=\boldsymbol{k}-\boldsymbol{p}$,

$$p\neq k: \qquad I_2 \xrightarrow[\mu\to 0]{} -\exp(i\tau_R)\frac{2k\nu}{q^2}\cos(qR), \tag{5.21a}$$

$$p=k: \qquad I_2 \xrightarrow[\mu\approx 0]{} -\exp(i\tau_R)\frac{2k\nu}{q^2}\cos(qR)$$
$$+\exp(i\Delta_R)f_c(\hat{\boldsymbol{p}},\boldsymbol{k})\left[-\Gamma(1+i\nu)\left(\frac{2k}{\mu}\right)^{i\nu}+\exp[i\nu\ln(2kR)]\right]. \tag{5.21b}$$

At least we find in appendix 5:

$$p\neq k: \qquad I_3=\exp(i\tau_R)\frac{2k\nu}{q^2}\cos(qR), \tag{5.22a}$$

$$p=k: \qquad I_3=\exp(i\tau_R)\frac{2k\nu}{q^2}\cos(qR)$$
$$+\exp[2i\nu\ln(2kR)]f_c(\hat{\boldsymbol{p}},\boldsymbol{k})[\exp(2i\tau_R)-\exp(i\tau_R)]. \tag{5.22b}$$

Inserting these results in (5.19*a*, *b*), we get in fact—after rather intricate cancellations—the expected results (5.18*a*, *b*).

Without the tail potential $W_R(r)$, i.e. with a sharp cut-off, (5.19*b*) shows that $I_3=0$, which leads to

$$t_R^{\text{sharp}}(\boldsymbol{p},\boldsymbol{k};k)_{p\neq k} \xrightarrow[R\approx\infty]{} \exp[i\nu\ln(2kR)]g_c(\boldsymbol{p},\boldsymbol{k})-\frac{2k\nu}{q^2}\cos(qR), \tag{5.23a}$$

$$t_R^{\text{sharp}}(\boldsymbol{p},\boldsymbol{k};k)_{p=k} \xrightarrow[R\approx\infty]{} \exp[2i\nu\ln(2kR)]f_c(\hat{\boldsymbol{p}},\boldsymbol{k})-\frac{2k\nu}{q^2}\cos(qR), \tag{5.23b}$$

in contrast to the result of Ford (1964), namely (5.18*a*). The first order term is for all $\boldsymbol{p}$ and $\boldsymbol{p}'$ equal to

$$t_R^{\text{sharp}}(\boldsymbol{p},\boldsymbol{p}';k)^{(1)}=v_R^{\text{sharp}}(\boldsymbol{p},\boldsymbol{p}')=\frac{2k\nu}{q^2}[1-\cos(qR)], \tag{5.24}$$

which cannot have (3.8*a*) as limit.

5.3. *Partial t-matrix*

The partial matrix $t_{R,l}(p,p';k)$ can be found for $p'=k$ and p arbitrary, when R is large but finite. Then (2.9) and (2.12*a*) from the 'short'-range theory show that

$$t_{R,l}(p,k;k)=-2m\int_0^\infty dr\, r^2 j_l(pr)V_R(r)\phi_{R,l}(r,k). \tag{5.25}$$

This can be written as $\lim_{\mu\to 0} J(\mu)$, where

$$J(\mu)=-2m\int_0^\infty \mathrm{d}r\, r^2 j_l(pr)\,\mathrm{e}^{-\mu r}V_c(r)\phi^{<}_{R,l}(r,k) \qquad (=J_1)$$

$$+2m\int_R^\infty \mathrm{d}r\, r^2 j_l(pr)\,\mathrm{e}^{-\mu r}V_c(r)\phi^{<}_{R,l}(r,k) \qquad (=J_2)$$

$$-2m\int_R^\infty \mathrm{d}r\, r^2 j_l(pr) W_R(r)\phi^{>}_{R,l}(r,k). \qquad (=J_3). \tag{5.26}$$

With (5.4a) and (5.7) the first integral is

$$J_1=\exp(\mathrm{i}\Delta_R)(-2m)\int_0^\infty \mathrm{d}r\, r^2 j_l(pr)\,\mathrm{e}^{-\mu r}V_c(r)\phi_{c,l}(r,k), \tag{5.27}$$

which compared with (3.18) and (3.19) yields:

$$p\neq k: \qquad J_1 \xrightarrow[\mu\to 0]{} \exp(\mathrm{i}\Delta_R) g_{c,l}(p,k), \tag{5.28a}$$

$$p=k: \qquad J_1 \xrightarrow[\mu\approx 0]{} \exp(\mathrm{i}\Delta_R+\mathrm{i}\sigma_l)\frac{1}{k}\,\mathrm{Im}\,\exp(\mathrm{i}\sigma_l)\Gamma(1+\mathrm{i}\nu)\left(\frac{2k}{\mu}\right)^{\mathrm{i}\nu}. \tag{5.28b}$$

Using the asymptotic forms of j_l and of $\phi^{<}_{R,l}$ (namely (5.4b)), we write the second integral in (5.26) as

$$J_2=-\frac{\nu}{p}\exp(\mathrm{i}\Delta_R+\mathrm{i}\sigma_l)\,\mathrm{Re}\,\exp(\mathrm{i}\sigma_l)$$

$$\times\int_R^\infty \mathrm{d}r\frac{1}{r}\exp[-\mu r+\mathrm{i}\nu\ln(2kR)]\{\exp[\mathrm{i}(k-p)r]-(-1)^l\exp[\mathrm{i}(k+p)r]\}. \tag{5.29}$$

By partial integrations and comparison with appendix (A.18), we get

$$p\neq k: \qquad J_2 \xrightarrow[\mu\approx 0]{} \mathrm{O}\left(\frac{1}{R}\right), \tag{5.30a}$$

$$p=k: \qquad J_2 \xrightarrow[\mu\approx 0]{} \exp(\mathrm{i}\Delta_R+\mathrm{i}\sigma_l)\frac{1}{k}\,\mathrm{Im}\,\exp(\mathrm{i}\sigma_l)\left[-\Gamma(1+\mathrm{i}\nu)\left(\frac{2k}{\mu}\right)^{\mathrm{i}\nu}+(2kR)^{\mathrm{i}\nu}\right]. \tag{5.30b}$$

With (5.5a) the third integral in (5.26) becomes:

$$J_3=\frac{\nu}{p}\exp(\mathrm{i}\Delta_R+\mathrm{i}\sigma_l)\,\mathrm{Re}\,\exp[\mathrm{i}\sigma_l+\mathrm{i}\nu\ln(2kR)]$$

$$\times\int_R^\infty \mathrm{d}r\, w_R(r)\exp[\mathrm{i}\tau_R(r)]\{\exp[\mathrm{i}(k-p)r]-(-1)^l\exp[\mathrm{i}(k+p)r]\}. \tag{5.31}$$

By partial integration and by recalling (5.5b): $\nu w_R(r)=\tau'_R(r)$, we get

$$p\neq k: \qquad J_3=\mathrm{O}\left(\frac{1}{R}\right), \tag{5.32a}$$

$$p=k: \qquad J_3=\exp(\mathrm{i}\Delta_R+\mathrm{i}\sigma_l)\frac{1}{k}\,\mathrm{Im}\,\exp(\mathrm{i}\sigma_l)[\exp(\mathrm{i}\Delta_R)-(2kR)^{\mathrm{i}\nu}]. \tag{5.32b}$$

Inserting these expressions in (5.26), we obtain at last

$$t_{R,l}(p,k;k)\xrightarrow[R\approx\infty]{}\begin{cases}\exp(\mathrm{i}\Delta_R)g_{c,l}(p,k), & p\neq k, \quad (5.33a)\\ \dfrac{1}{2\mathrm{i}k}[\exp(2\mathrm{i}\delta_l)-1], & p=k. \quad (5.33b)\end{cases}$$

(Note. The sum of (5.30a) and (5.32a) is of $O(R^{-2})$.) With a treatment like (4.24) we may write (5.33b) in the form

$$t_{R,l}(k,k;k)=f_{R,l}(k)\xrightarrow[R\approx\infty]{}\exp(2\mathrm{i}\Delta_R)f_{c,l}(k), \tag{5.33c}$$

i.e. in accordance with (5.18b).

The results (5.33) for $l=0$ have been obtained by Ford (1964), in spite of his use of the sharp cut-off.

6. Conclusion

To summarise, our conjecture is the following. With

$$\lim_{\alpha\to 0} V_\alpha(r)=V_c(r), \qquad V_\alpha(r) \text{ continuous}, \tag{6.1}$$

one will get:

$$p'\neq k\neq p: \quad t_\alpha \xrightarrow[\alpha\to 0]{} t_c(\boldsymbol{p},\boldsymbol{p}';k), \tag{6.2}$$

$$p'=k\neq p: \quad t_\alpha=g_\alpha \xrightarrow[\alpha\approx 0]{} g_c(\boldsymbol{p},\boldsymbol{k})\exp(\mathrm{i}\Delta_\alpha), \tag{6.3}$$

$$p'=k=p: \quad t_\alpha=g_\alpha=f_\alpha \xrightarrow[\alpha\approx 0]{} f_c(\hat{\boldsymbol{p}},\boldsymbol{k})\exp(2\mathrm{i}\Delta_\alpha). \tag{6.4}$$

The same relations hold for the partial amplitudes. The Coulomb quantities t_c, g_c and f_c are defined for $p'\neq k\neq p$, $p\neq k$ and $p=k$, respectively. The absolute values of t_c and g_c, however, exist on the total energy shell, and are given by

$$\lim_{\epsilon\to 0}\lim_{\substack{p'\to k\\ p\to k}}|t_c(\boldsymbol{p},\boldsymbol{p}';k_\epsilon)|=|f_c|.|\Gamma(1+\mathrm{i}\nu)|^2, \tag{6.5}$$

$$\lim_{\epsilon\to 0}\lim_{p\to k}|g_{c\epsilon}(\boldsymbol{p},\boldsymbol{k})|=|f_c|.|\Gamma(1+\mathrm{i}\nu)|. \tag{6.6}$$

Examples of V_α and Δ_α are:

$$\alpha=\mu: \qquad V_\mu=-\frac{\nu k}{m}\frac{1}{r}\mathrm{e}^{-\mu r}, \tag{6.7a}$$

$$\Delta_\mu=\nu\left(\ln\frac{2k}{\mu}-\gamma\right); \tag{6.7b}$$

$$\alpha=\frac{1}{R}: \qquad V_R=-\frac{\nu k}{m}\begin{cases}1/r, & r\leqslant R,\\ w_R(r), & r\geqslant R,\end{cases} \tag{6.8a}$$

$$\Delta_R=\nu\left(\ln(2kR)+\int_R^\infty \mathrm{d}r\, w_R(r)\right). \tag{6.8b}$$

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Appendix 1

The Coulomb scattering amplitude (3.3*a*) can be written (with $\xi = \hat{\boldsymbol{p}}.\hat{\boldsymbol{k}}$)

$$f_c(\xi, k) = \frac{\nu}{k} e^{2i\sigma_0} 2^{-i\nu} (1-\xi)^{i\nu-1} = \frac{d}{d\xi} \frac{i}{k} e^{2i\sigma_0} 2^{-i\nu} (1-\xi)^{i\nu} = \frac{d}{d\xi} \frac{i}{k} e^{2i\sigma_0} 2^{-i\nu} \sum_{l=0}^{\infty} a_l P_l(\xi). \tag{A.1}$$

The expansion of $(1-\xi)^{i\nu}$ can be inverted, which gives

$$a_l = \frac{2l+1}{2^l l!} \int_{-1}^{+1} d\xi (1-\xi)^{i\nu} \frac{d^l}{d\xi^l} (\xi^2 - 1)^l = 2^{i\nu} (2l+1) \frac{\Gamma(l-i\nu)}{\Gamma(-i\nu)} \frac{\Gamma(1+i\nu)}{\Gamma(l+2+i\nu)}, \tag{A.2}$$

(by partial integrations). Hence

$$f_c(\xi, k) = \frac{\nu}{k} \sum_{l=1}^{\infty} (2l+1) \frac{\Gamma(l-i\nu)}{\Gamma(l+2+i\nu)} \frac{dP_l(\xi)}{d\xi}, \tag{A.3}$$

as in Magnus *et al* (1966), with the Gegenbauer polynomials $C_{l-1}^{3/2}(\xi) = dP_l(\xi)/d\xi$.

The lth term is of $O(l^{-1}) \times O(l^{1/2}) = O(l^{-1/2})$ for $l \gg 1$ (Magnus *et al* 1966).† Observing that

$$\exp(2i\sigma_{l-1}) - \exp(2i\sigma_{l+1}) = \frac{\Gamma(l-i\nu)}{\Gamma(l+i\nu)} - \frac{\Gamma(l+2-i\nu)}{\Gamma(l+2+i\nu)} = \frac{\Gamma(l-i\nu)}{\Gamma(l+2+i\nu)} 2i\nu(2l+1), \tag{A.4}$$

we can write (A.3) in the form

$$f_c(\xi, k) = \frac{1}{2ik} \sum_{l=1}^{\infty} [\exp(2i\sigma_{l-1}) - \exp(2i\sigma_{l+1})] \frac{dP_l(\xi)}{d\xi}. \tag{A.5}$$

Appendix 2

An expansion of $(1-\xi)^{-1}$ is of course given in the first-order term of (A.5). It is, however, easily obtained in the following way:

$$1 + \xi = P_0(\xi) + P_1(\xi) = \sum_{l=1}^{\infty} [P_{l-1}(\xi) - P_{l+1}(\xi)] = \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} (1-\xi^2) \frac{dP_l(\xi)}{d\xi}, \tag{A.6}$$

according to a formula for Legendre polynomials. Hence—for $\xi^2 \neq 1$

$$\frac{1}{1-\xi} = \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} \frac{dP_l(\xi)}{d\xi} = \sum_{l=1}^{\infty} \left(\frac{1}{l} + \frac{1}{l+1}\right) \frac{dP_l(\xi)}{d\xi}$$

$$= \sum_{l=1}^{\infty} [\psi(l+2) - \psi(l)] \frac{dP_l(\xi)}{d\xi}, \tag{A.7}$$

where $\psi(l)$ is given in (3.23*b*).

† *Note added in proof.* There is numerical evidence that the series (A.3) is also divergent, like (3.16) (J Midtdal 1978, private communication).

Appendix 3

To evaluate the integral (3.18*b*), where $\phi_{c,l}$ is given in (3.13*a*), we use a contour integral representation of the confluent hypergeometric function F in $\phi_{c,l}$, and get

$$
\begin{aligned}
g_{c\epsilon,l}(p,k) &= 2k\nu\int_0^\infty \mathrm{d}r\, r\, \mathrm{e}^{-\epsilon r} j_l(pr)\, \mathrm{e}^{\nu\pi/2}\frac{\Gamma(l+1-\mathrm{i}\nu)}{(2l+1)!}\mathrm{e}^{\mathrm{i}kr}(2kr)^l \\
&\quad\times(2l+1)!\frac{\Gamma(-l-\mathrm{i}\nu)}{\Gamma(l+1-\mathrm{i}\nu)}\frac{1}{2\pi\mathrm{i}}\oint_{(0,1)}\mathrm{d}t\, \mathrm{e}^{-2\mathrm{i}krt}t^{l-\mathrm{i}\nu}(t-1)^{l+\mathrm{i}\nu} \\
&= \nu(2k)^{l+1}(2p)^{-1/2}\,\mathrm{e}^{\nu\pi/2}\Gamma(-l-\mathrm{i}\nu)\pi^{1/2}\oint \mathrm{d}t\, t^{l-\mathrm{i}\nu}(t-1)^{l+\mathrm{i}\nu} \\
&\quad\times\int_0^\infty \mathrm{d}r\, \mathrm{e}^{-[\epsilon+\mathrm{i}k(2t-1)]r}r^{l+1/2}J_{l+1/2}(pr).
\end{aligned}
\tag{A.8}
$$

With the Laplace-transform (Magnus *et al* 1966) we get

$$
g_{c\epsilon,l}(p,k) = -\nu\left(-\frac{p}{k}\right)^l \mathrm{e}^{\nu\pi/2}l!\,\Gamma(-l-\mathrm{i}\nu)C(\epsilon), \tag{A.9}
$$

where

$$
C(\epsilon) = \frac{1}{2\pi\mathrm{i}}\oint_{(0,2k)}\mathrm{d}z\frac{z^{l-\mathrm{i}\nu}(z-2k)^{l+\mathrm{i}\nu}}{(z-p_+)^{l+1}(z-p_-)^{l+1}}, \tag{A.10}
$$

with

$$
p_\pm = k \pm p + \mathrm{i}\epsilon. \tag{A.11}
$$

The path of integration around 0 and $2k$ may be deformed to two clockwise paths around the poles p_+ and p_-, and a circle with infinite radius. The contributions from the pair of parallel paths extending from p_+ and p_- to infinity cancel. The contribution from the large circle vanishes because the integrand is of $O(z^{-2})$. Hence the only contributions are from the poles p_+ and p_-, so that

$$
g_{c\epsilon,l}(p,k) = \mathrm{e}^{\nu\pi/2}2\mathrm{i}\nu\Gamma(-l-\mathrm{i}\nu)\left(-\frac{p}{k}\right)^l \mathrm{Im}\left[\frac{\partial^l}{\partial p^l}\frac{(p+k+\mathrm{i}\epsilon)^{l-\mathrm{i}\nu}(p-k+\mathrm{i}\epsilon)^{l+\mathrm{i}\nu}}{(p+q)^{l+1}}\right]_{q=p}. \tag{A.12}
$$

When $p \neq k$, we can put $\epsilon = 0$ in (A.12) to get $g_{c,l}(p,k)$.

When $p = k$, the factor $(p-k+\mathrm{i}\epsilon)^{l+\mathrm{i}\nu}$ vanishes with ϵ, except when differentiated l times. Hence

$$
\begin{aligned}
g_{c\epsilon,l}(k,k) &\xrightarrow[\epsilon\approx 0]{} \mathrm{e}^{\nu\pi/2}(-1)^l\mathrm{i}\nu\Gamma(-l-\mathrm{i}\nu)\frac{1}{k}\mathrm{Im}(2k)^{-\mathrm{i}\nu}(l+\mathrm{i}\nu)(l-1+\mathrm{i}\nu)\dots(1+\mathrm{i}\nu)(\mathrm{i}\epsilon)^{\mathrm{i}\nu} \\
&= -\frac{1}{k}\Gamma(l+1-\mathrm{i}\nu)\,\mathrm{Im}\frac{\Gamma(1-\mathrm{i}\nu)}{\Gamma(l+1-\mathrm{i}\nu)}\left(\frac{2k}{\epsilon}\right)^{-\mathrm{i}\nu} \\
&= \frac{1}{k}\exp(\mathrm{i}\sigma_l)\,\mathrm{Im}\exp(\mathrm{i}\sigma_l)\Gamma(1+\mathrm{i}\nu)\left(\frac{2k}{\epsilon}\right)^{\mathrm{i}\nu}.
\end{aligned}
\tag{A.13}
$$

Appendix 4

Consider the integral in (5.19*b*)

$$I_2=\frac{m}{2\pi}\exp(\mathrm{i}\Delta_R)\int_{(r>R)}\mathrm{d}^3\boldsymbol{r}\,\exp(-\mathrm{i}\boldsymbol{p}.\boldsymbol{r}-\mu r)V_c(r)\psi_c(\boldsymbol{r},\boldsymbol{k}). \tag{A.14}$$

When $kR\gg 1$, the asymptotic forms (3.2*b*) and (3.4) (slightly modified) shall be used, so that

$$\begin{aligned}
I_2=&-\exp(\mathrm{i}\Delta_R)\frac{k\nu}{2\pi}\int_{(r>R)}\mathrm{d}^3\boldsymbol{r}\,\exp(-\mathrm{i}\boldsymbol{p}.\boldsymbol{r})\frac{\exp(-\mu r)}{r}\\
&\times\left(\exp[\mathrm{i}\boldsymbol{k}.\boldsymbol{r}-\mathrm{i}\nu\ln(2kr)]+f_c(\hat{\boldsymbol{r}},\boldsymbol{k})\frac{1}{r}\exp[\mathrm{i}kr+\mathrm{i}\nu\ln(2kr)]\right)\\
=&-\exp(\mathrm{i}\Delta_R)\frac{k\nu}{2\pi}\int_R^\infty\mathrm{d}r\,r\exp(-\mu r)\int\mathrm{d}\Omega_r\Big(\exp[-\mathrm{i}\nu\ln(2kr)]\exp(\mathrm{i}\boldsymbol{q}.\boldsymbol{r})\\
&+\frac{2\pi}{\mathrm{i}pr}[\exp(\mathrm{i}pr)\delta(\Omega_r+\Omega_p)-\exp(-\mathrm{i}pr)\delta(\Omega_r-\Omega_p)]f_c(\hat{\boldsymbol{r}},\boldsymbol{k})\frac{\exp(\mathrm{i}kr)}{r}(2kr)^{\mathrm{i}\nu}\Big)\\
=&-\exp(\mathrm{i}\Delta_R)2k\nu\Big(\frac{1}{q}\int_R^\infty\mathrm{d}r\,\sin(qr)\exp[-\mu r-\mathrm{i}\nu\ln(2kr)]\\
&+\frac{k}{\mathrm{i}p}f_c(-\hat{\boldsymbol{p}},\boldsymbol{k})\int_R^\infty\mathrm{d}r\,\exp[-\mu r+\mathrm{i}(k+p)r](2kr)^{-1+\mathrm{i}\nu}\\
&-\frac{k}{\mathrm{i}p}f_c(\hat{\boldsymbol{p}},\boldsymbol{k})\int_R^\infty\mathrm{d}r\,\exp[-\mu r+\mathrm{i}(k-p)r](2kr)^{-1+\mathrm{i}\nu}\Big),
\end{aligned} \tag{A.15}$$

where $\boldsymbol{q}=\boldsymbol{k}-\boldsymbol{p}$. From partial integration we get to $\mathrm{O}(R^0)$:

$$I_2\xrightarrow[\mu\approx 0]{}-\exp[\mathrm{i}\Delta_R-\mathrm{i}\nu\ln(2kR)]\frac{2k\nu}{q^2}\cos(qR)+\exp(\mathrm{i}\Delta_R)f_c(\hat{\boldsymbol{p}},\boldsymbol{k})D(\mu), \tag{A.16}$$

where

$$D(\mu)=\frac{2\nu k^2}{\mathrm{i}p}\int_R^\infty\mathrm{d}r\,\exp[-\mu r+\mathrm{i}(k-p)r](2kr)^{-1+\mathrm{i}\nu}. \tag{A.17}$$

When $p\neq k$, partial integration shows that $D(\mu)=\mathrm{O}(R^{-1})$. When $p=k$, we write

$$\begin{aligned}
D(\mu)&=-\mathrm{i}\nu\left(\frac{2k}{\mu}\right)^{\mathrm{i}\nu}\left\{\int_0^\infty-\int_0^{\mu R}\right\}\mathrm{d}x\,\mathrm{e}^{-x}x^{-1+\mathrm{i}\nu}\\
&=-\mathrm{i}\nu\left(\frac{2k}{\mu}\right)^{\mathrm{i}\nu}\left[\Gamma(\mathrm{i}\nu)-\mathrm{e}^{-\mu R}\left(\frac{(\mu R)^{\mathrm{i}\nu}}{\mathrm{i}\nu}+\cdots\right)\right]\\
&\xrightarrow[\mu\approx 0]{}-\left(\frac{2k}{\mu}\right)^{\mathrm{i}\nu}\Gamma(1+\mathrm{i}\nu)+(2kR)^{\mathrm{i}\nu}.
\end{aligned} \tag{A.18}$$

Here we have used $\mathrm{i}\nu=\lim_{\epsilon\to 0}(\mathrm{i}\nu+\epsilon)$, because (A.17) exists. We also point out that when $\mu\to 0$, R is finite. The limit $R\to\infty$ shall be taken at the very end, according to our procedure.

Appendix 5

Consider I_3 in (5.19*b*):

$$I_3 = -\frac{m}{2\pi}\int_{(r>R)} \mathrm{d}^3\boldsymbol{r}\, \exp(-\mathrm{i}\boldsymbol{p}.\boldsymbol{r})W_R(r)\psi_R^{\gtrless}(\boldsymbol{r}, \boldsymbol{k}). \tag{A.19}$$

Substituting (5.9) and (3.4), we get

$$\begin{aligned}
I_3 &= \frac{\nu}{2\pi}\int \mathrm{d}^3\boldsymbol{r}\, \exp(-\mathrm{i}\boldsymbol{p}.\boldsymbol{r})w_R(r)\exp(\mathrm{i}\tau_R)\Big(\exp[\mathrm{i}\boldsymbol{k}.\boldsymbol{r} - \mathrm{i}\tau_R(r)] \\
&\quad + \exp[2\mathrm{i}\nu\ln(2kR)]f_c(\hat{\boldsymbol{r}}, \boldsymbol{k})\frac{1}{r}\exp[\mathrm{i}kr + \mathrm{i}\tau_R(r)]\Big) \\
&= \exp(\mathrm{i}\tau_R)\frac{\nu k}{2\pi}\int_R^\infty \mathrm{d}r\, r^2 w_R(r)\int \mathrm{d}\Omega_r\Big(\exp[-\mathrm{i}\tau_R(r)]\exp(\mathrm{i}\boldsymbol{q}.\boldsymbol{r}) \\
&\quad + \exp[2\mathrm{i}\nu\ln(2kR)]\frac{2\pi}{\mathrm{i}pr}[\exp(\mathrm{i}pr)\delta(\Omega_r + \Omega_p) \\
&\quad - \exp(-\mathrm{i}pr)\delta(\Omega_r - \Omega_p)]f_c(\hat{\boldsymbol{r}}, \boldsymbol{k})\frac{1}{r}\exp(\mathrm{i}kr + \mathrm{i}\tau_R(r))\Big) \\
&= \exp(\mathrm{i}\tau_R)\Big(\frac{2\nu k}{q}\int_R^\infty \mathrm{d}r\, r w_R(r)\exp[-\mathrm{i}\tau_R(r)]\sin(qr) \\
&\quad - \exp[2\mathrm{i}\nu\ln(2kR)]f_c(-\hat{\boldsymbol{p}}, \boldsymbol{k})\frac{k}{p}\int_R^\infty \mathrm{d}r\, \exp[\mathrm{i}(k+p)r]\frac{\mathrm{d}}{\mathrm{d}r}\exp[\mathrm{i}\tau_R(r)] \\
&\quad + \exp[2\mathrm{i}\nu\ln(2kR)]f_c(\hat{\boldsymbol{p}}, \boldsymbol{k})\frac{k}{p}\int_R^\infty \mathrm{d}r\, \exp[\mathrm{i}(k-p)r]\frac{\mathrm{d}}{\mathrm{d}r}\exp[\mathrm{i}\tau_R(r)]\Big),
\end{aligned} \tag{A.20}$$

(recall (5.6*b*)). Using partial integrations, we get, apart from terms of $O(R^{-1})$, when $p \neq k$:

$$\begin{aligned}
I_3(p \neq k) &= \exp(\mathrm{i}\tau_R)\frac{2\nu k}{q}\Big[-r w_R(r)\exp(-\mathrm{i}\tau_R(r))\frac{\cos(qR)}{q}\Big]_R^\infty \\
&= \exp(\mathrm{i}\tau_R)\frac{2\nu k}{q^2}\cos(qR),
\end{aligned} \tag{A.21}$$

where (5.1*b*) has been used. But when $p = k$, we get

$$I_3(p = k) = \exp(\mathrm{i}\tau_R)\frac{2\nu k}{q^2}\cos(qR) + \exp[\mathrm{i}\tau_R + 2\mathrm{i}\nu\ln(2kR)]f_c(\hat{\boldsymbol{p}}, \boldsymbol{k})[\exp(\mathrm{i}\tau_R) - 1]. \tag{A.22}$$

Appendix 6

An integral equation for $g_c(\boldsymbol{p}, \boldsymbol{k})$ can be established in the following way. With a screened potential the equations (2.3) and (2.5*a*) show that

$$\frac{1}{v(\boldsymbol{p},\boldsymbol{k})}\left(g(\boldsymbol{p},\boldsymbol{k})-\frac{1}{2\pi^2}\int \mathrm{d}^3\boldsymbol{q}\, v(\boldsymbol{p},\boldsymbol{q})\frac{1}{q^2-k_\epsilon^2}g(\boldsymbol{q},\boldsymbol{k})\right)$$
$$=\frac{1}{v(\boldsymbol{p}_0,\boldsymbol{k})}\left(g(\boldsymbol{p}_0,\boldsymbol{k})-\frac{1}{2\pi^2}\int \mathrm{d}^3\boldsymbol{q}\, v(\boldsymbol{p}_0,\boldsymbol{q})\frac{1}{q^2-k_\epsilon^2}g(\boldsymbol{q},\boldsymbol{k})\right), \qquad \text{(A.23)}$$

where $\boldsymbol{p}_0$ is arbitrary. Hence

$$\frac{g(\boldsymbol{p},\boldsymbol{k})}{g(\boldsymbol{p}_0,\boldsymbol{k})}=\frac{v(\boldsymbol{p},\boldsymbol{k})}{v(\boldsymbol{p}_0,\boldsymbol{k})}+\frac{1}{2\pi^2}\int \mathrm{d}^3\boldsymbol{q}\left(v(\boldsymbol{p},\boldsymbol{q})-\frac{v(\boldsymbol{p},\boldsymbol{k})}{v(\boldsymbol{p}_0,\boldsymbol{k})}v(\boldsymbol{p}_0,\boldsymbol{q})\right)\frac{1}{q^2-k_\epsilon^2}\frac{g(\boldsymbol{q},\boldsymbol{k})}{g(\boldsymbol{p}_0,\boldsymbol{k})}. \qquad \text{(A.24)}$$

As $g(\boldsymbol{p},\boldsymbol{k})\to g_c(\boldsymbol{p},\boldsymbol{k})\exp[\mathrm{i}\Delta(k)]$ when $v\to v_c$, equation (A.24) is valid also for the Coulomb case, because the phases $\Delta(k)$ cancel.

Alternatively, (A.24) can be deduced from the following equation, which is valid also for Coulomb scattering, (given by Kolsrud 1977),

$$|\psi_{\boldsymbol{k}}\rangle=|\boldsymbol{k}\rangle\frac{\langle c|\psi_{\boldsymbol{k}}\rangle}{\langle c|\boldsymbol{k}\rangle}+\left(1-\frac{|\boldsymbol{k}\rangle\langle c|}{\langle c|\boldsymbol{k}\rangle}\right)\frac{1}{E_k+\mathrm{i}\epsilon-H_0}V|\psi_{\boldsymbol{k}}\rangle \qquad \text{(A.25)}$$

where $\langle c|$ is arbitrary. Obviously $(E_k-H_0)|\psi_{\boldsymbol{k}}\rangle=V|\psi_{\boldsymbol{k}}\rangle$. Choosing $\langle c|=\langle\boldsymbol{p}_0|V$, and multiplying (A.25) by $\langle\boldsymbol{p}|V$, we get

$$\frac{\langle\boldsymbol{p}|V|\psi_{\boldsymbol{k}}\rangle}{\langle\boldsymbol{p}_0|V|\psi_{\boldsymbol{k}}\rangle}=\frac{\langle\boldsymbol{p}|V|\boldsymbol{k}\rangle}{\langle\boldsymbol{p}_0|V|\boldsymbol{k}\rangle}+\int \mathrm{d}^3\boldsymbol{q}\left(\langle\boldsymbol{p}|V|\boldsymbol{q}\rangle-\frac{\langle\boldsymbol{p}|V|\boldsymbol{k}\rangle}{\langle\boldsymbol{p}_0|V|\boldsymbol{k}\rangle}\langle\boldsymbol{p}_0|V|\boldsymbol{q}\rangle\right)\frac{1}{E_k+\mathrm{i}\epsilon-E_q}\frac{\langle\boldsymbol{q}|V|\psi_{\boldsymbol{k}}\rangle}{\langle\boldsymbol{p}_0|V|\psi_{\boldsymbol{k}}\rangle}, \qquad \text{(A.26)}$$

which equals (A.24) (recall (2.2)). In a similar way $t(\boldsymbol{p},\boldsymbol{p}';k)$ may be treated.

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